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Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups.

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Abstract

We give a stratification of the GIT quotient of the Grassmannian $G_{2,n}$ modulo the normaliser of a maximal torus of $SL_n(k)$ with respect to the ample generator of the Picard group of $G_{2,n}$. We also prove that the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$.

Keywords: GIT quotient, line bundle, simple reflection.

Introduction

Let k be an algebraically closed field. Consider the action of a maximal torus T of $SL_n(k)$ on the Grassmannian $G_{r,n}$ of r- dimensional vector subspaces of an n- dimensional vector space over k. Let N denote the normaliser of T in $SL_n(k)$. Let $\mathcal{L}_r$ denote the ample generator of the Picard group of $G_{r,n}$. Let $W = N/T$ denote the Weyl group of $SL_n(k)$ with respect to T.

In [5], it is shown that the semi-stable points of $G_{r,n}$ with respect to the T-linearised line bundle $\mathcal{L}_r$ is same as the stable points if and only if r and n are co-prime.

In this paper, we describe all the semi-stable points of $G_{r,n}$ with respect to $\mathcal{L}_r$. In this connection, we prove the following result:
First, we introduce some notation needed for the statement of the theorem.

Let $\mathfrak{h}_j$ be a Cartan subalgebra of $\mathfrak{sl}_{j+1}$, $\mathbb{P}(\mathfrak{h}_j)$ be the projective space and $R_j \subseteq \mathfrak{h}_j^*$ be the root system. Let V_j be the open subset of $\mathbb{P}(\mathfrak{h}_j)$ defined by

$$V_j := \{x \in \mathbb{P}(\mathfrak{h}_j) : \alpha(x) \neq 0, \forall \alpha \in R_j\}.$$

Here, the Weyl group of $\mathfrak{sl}_{j+1}$ is S_{j+1}, and $\mathfrak{h}_j$ is the standard representation of S_{j+1}.

With this notation, taking $m = \lceil \frac{n-1}{2} \rceil$ (for this notation, see lemma 1.6) and $t = [\frac{n-1}{2}]$ we have

Theorem: ${}_N\backslash\backslash G_{2,n}^{ss}(\mathcal{L}_2)$ has a stratification $\bigcup_{i=0}^{t} C_i$ where $C_0 = {}_{S_{m+1}}\backslash\mathbb{P}(\mathfrak{h}_m)$, and $C_i = {}_{S_{i+m+1}}\backslash V_{i+m}$.

On the other hand, the GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for any ample line bundle on $GL_{n+1}(k)/B_{n+1}$ and $GL_n(k)/B_n$ are both birational varieties. So, it is a natural question to ask whether the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$. We give an affirmative answer to this question. For a more precise statement, see theorem 5.2. In this connection, we also prove that the action of the Weyl group S_{n+1} on the quotient is given by the standard representation. For a more precise statement, see corollary 5.4.

Section 1 consists of preliminary notation and some combinatorial lemmas about minuscule weights.

In section 2, we describe all Schubert cells in $G_{r,n}$ admitting semi-stable points.

In section 3, we describe the action of the Weyl group W on ${}_T\backslash\backslash G_{r,n}^{ss}(\mathcal{L}_r)$.

In section 4, we describe a stratification of ${}_N\backslash\backslash G_{2,n}^{ss}(\mathcal{L}_2)$.

In section 5, we obtain $GL_n(k)/B_n$ as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable line bundle on $GL_{n+1}(k)/B_{n+1}$.

1 Preliminary notation and some combinatorial Lemmas

This section consists of preliminary notation and some combinatorial lemmas about minuscule weights. Let G be a reductive Chevalley group over an algebraically closed field k. Let T be a maximal torus of the commutator subgroup $[G, G]$, B a Borel subgroup of G containing T and U be the unipotent radical of B. Let N be the normaliser of T in $[G, G]$. Let $W = N/T$ be Weyl group of $[G, G]$ with respect to T and R denote the set of roots with respect to T, R^+ positive roots with respect to B. Let U_α denote the one dimentional

T-stable subgroup of G corresponding to the root α and let $S = \{\alpha_1, \cdots \alpha_l\} \in R^+$ denote the set of simple roots. For a subset $I \subseteq S$ denote $W^I = \{w \in W | w(\alpha) > 0,\ \alpha \in I\}$. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of T (resp. one parameter subgroups of T). Let $E_1 := X(T) \otimes \mathbb{R}$, $E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle .,. \rangle : E_1 \times E_2 \longrightarrow \mathbb{R}$ be the canonical non-degenerate bilinear form. Choose λ_j's in E_2 such that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for all i. Let $\overline{C(B)} := \mathbb{R}_{\geq 0}$ - span of the λ_i's . Let $\check{\alpha} \in Y(T)$ be as in page-19 of [1]. We also have $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\alpha \in R$ and $\chi \in E_1$. Set $s_i = s_{\alpha_i}\ \forall\ i = 1, 2 \cdots l$. Let $\{\omega_i : i = 1, 2 \cdots l\} \subset E_1$ be the fundamental weights; i.e. $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2 \cdots l$.

We now prove some elementary lemmas about minuscule weights. For notation, we refer to [7].

Lemma 1.1. *Let I be any nonempty subset of S, and let μ be a weight of the form $\sum_{\alpha_i \in I} m_i\alpha_i - \sum_{\alpha_i \notin I} m_i\alpha_i$, where $m_i \in \mathbb{Q}$ for all i, $1 \leq i \leq l$; $m_i > 0$ for all $\alpha_i \in I$ and $m_i \geq 0$ for all $\alpha_i \in S \setminus I$. Then there is an $\alpha \in I$ such that $s_\alpha(\mu) < \mu$.*

Proof. Since $s_\alpha(\mu) = \mu - \langle \mu, \check{\alpha} \rangle \alpha$, we need to find an $\alpha \in I$ such that $\langle \mu, \check{\alpha} \rangle > 0$. This follows because the Cartan matrix $(\langle \alpha_i, \check{\alpha}_j \rangle)_{i,j}$ is positive definite, so we can find an $\alpha \in I$ such that $\langle \sum_{\alpha_i \in I} m_i\alpha_i, \check{\alpha} \rangle > 0$. Now we know that for any $\alpha_i, \alpha_j \in S$, $i \neq j$, $\langle \alpha_i, \check{\alpha}_j\ rangle \leq 0$. Hence, $\langle \sum_{\alpha_i \notin I} m_i\alpha_i, \check{\alpha} \rangle \leq 0$ for this $\alpha \in I$. Thus $\langle \mu, \check{\alpha} \rangle > 0$. This proves the lemma. □

Lemma 1.2. *Let λ be any dominant weight and let $I = \{\alpha \in S : \langle \lambda, \check{\alpha} \rangle = 0\}$. Let $w_1, w_2 \in W^I$ be such that $w_1(\lambda) = w_2(\lambda)$. Then $w_1 = w_2$.*

Proof. See [1] and [2]. □

In the rest of this section, ω will denote a minuscule weight and $I := \{\alpha \in S : \langle \omega, \check{\alpha} \rangle = 0\}$

Lemma 1.3. *Let $\alpha \in S$ and $\tau \in W$ such that $l(s_\alpha\tau) = l(\tau) + 1$ and $s_\alpha\tau \in W^I$, then $\tau \in W^I$; $s_\alpha\tau(\omega) = \tau(\omega) - \alpha$.*

Proof. The proof of the first part of the lemma is clear. Now $s_\alpha\tau(\omega) = \tau(\omega) - \langle \tau(\omega), \check{\alpha} \rangle \alpha$. Since the pairing $\langle .,. \rangle$ is W-invariant, $\langle \tau(\omega), \check{\alpha} \rangle = \langle \omega, \tau^{-1}\check{\alpha} \rangle$. Again since $l(s_\alpha\tau) = l(\tau) + 1$, we have $\tau^{-1}\alpha > 0$. Let $\tau^{-1}\check{\alpha} = \sum_{i=1}^{l} m_i\check{\alpha}_i$, $m_i \in \mathbb{Z}_{\geq 0}$. Now, if $\langle \omega, \tau^{-1}\check{\alpha} \rangle = 0$, then $m_i > 0 \Rightarrow \langle \omega, \tau^{-1}\check{\alpha}_i \rangle = 0$ for $1 \leq i \leq l$. This gives a contradiction, since $s_\alpha\tau \in W^I$ and $s_\alpha\tau(\tau^{-1}\alpha) = s_\alpha(\alpha) < 0$. Thus, $\langle \omega, \tau^{-1}\check{\alpha} \rangle = 1$. Hence the lemma is proved. □

Corollary 1.4. 1. *For any $w \in W^I$, the number of times that s_i, $1 \leq i \leq n-1$ appears in a reduced expression of w = (coefficient of α_i in ω) $-$ (coefficient of α_i in $w(\omega)$) and hence it is independent of the reduced expression of w.*
2. *Let $w \in W^I$ and let $w = s_{i_1}.s_{i_2} \ldots s_{i_k} \in W^I$ be a reduced expression. Then $w(\omega) = \omega - \sum_{j=1}^{k} \alpha_{i_j}$. and $l(w) = ht(\omega - w(\omega))$.*

Proof. Follows from Lemma 1.3. □

Lemma 1.5. *Let $w = s_{i_1}s_{i_2}\dots s_{i_k} \in W$ such that $ht(\omega - s_{i_1}s_{i_2}\dots s_{i_k}(\omega)) = k$ then $w \in W^I$ and $l(w) = k$.*

Proof. This follows from the corollary 1.4. □

Lemma 1.6. *Let $\omega = \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Q}_{\geq 0}$ be a minuscule weight. Let $I = \{\alpha \in S : \langle \omega, \check{\alpha}\rangle = 0\}$. Then, there exist a unique $w \in W^I$ such that $w(\omega) = \sum_{i=1}^{l}(m_i - \lceil m_i \rceil)\alpha_i$ where for any real number x,*

$$\lceil x \rceil := \begin{cases} x & \text{if } x \text{ is an integer} \\ [x]+1 & \text{otherwise} \end{cases}$$

Proof. Using lemma 1.1 and the fact that ω is minuscule we can find a sequence $s_{i_k}, s_{i_{k-1}}, \cdots, s_{i_1}$ of simple reflections in W such that for each j, $2 \leq j \leq k+1$, coefficient of α_{i_j} in $s_{i_{j-1}}.s_{i_{j-2}}\dots s_{i_1}(\omega_r)$ is positive and $(s_{i_k}.s_{i_{k-1}}\dots s_{i_1}(\omega_r)) = \omega_r - \sum_{j=1}^{k}\alpha_{i_j}$ for each j, $1 \leq j \leq k$. The existence part of the lemma follows from here. The uniqueness follows from lemma 1.2. □

Lemma 1.7. *Let $\omega = \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Q}_{\geq 0}$ be a minuscule weight. Let $I = \{\alpha \in S : \langle \omega, \check{\alpha}\rangle = 0\}$. Then, there exist a unique $\tau \in W^I$ such that $\tau(\omega) = \sum_{i=1}^{l}(m_i - [m_i])\alpha_i$.*

Proof. Proof is similar to that of lemma 1.6. □

Now onwards, we say that for two elements w and τ in W, $w \leq \tau$ if $l(\tau) = l(w) + l(\tau w^{-1})$.

Lemma 1.8. *Let ω and I be as in the lemma 1.6 and $\tau, \sigma \in W^I$. Then $\tau(\omega) \leq \sigma(\omega) \Leftrightarrow \sigma \leq \tau$.*

Proof. The proof is by induction on $ht(\sigma(\omega) - \tau(\omega))$ which is a non-negative integer.
$\underline{ht(w(\sigma\omega) - \tau(\omega)) = 1}$: This means $\sigma(\omega) = \tau(\omega) + \alpha$ for some $\alpha \in S$. Applying s_α on both the sides of this equation, we have,

$$\begin{aligned} & s_\alpha\sigma(\omega) = -\alpha + s_\alpha\tau(\omega) \\ \implies & \tau(\omega) - \langle \omega, \sigma^{-1}\check{\alpha}\rangle\alpha = -2\alpha + \tau(\omega) - \langle \omega, \tau^{-1}\check{\alpha}\rangle\alpha \\ \implies & \langle \omega, \sigma^{-1}\check{\alpha}\rangle = 2 + \langle \omega, \tau^{-1}\check{\alpha}\rangle \end{aligned}$$

Since ω is minuscule, we get $\langle \omega, \sigma^{-1}\check{\alpha}\rangle = 1$ and $\langle \omega, \tau^{-1}\check{\alpha}\rangle = -1$. This implies, by the lemma 1.5, that $l(s_\alpha\sigma) = l(w) + 1$ and $s_\alpha w \in W^I$. Now, we have $s_\alpha\sigma(\omega) = \tau(\omega)$. Hence, by lemma 1.2, we get $\tau = s_\alpha\sigma$ with $l(\tau) = l(\sigma) + 1$. Thus the result follows in this case.
Let us assume that the result is true for $ht(\sigma(\omega) - \tau(\omega)) \leq m - 1$.
$\underline{ht(\sigma(\omega) - \tau(\omega)) = m}$: Let $\sigma(\omega) - \tau(\omega) = \sum_{\alpha_i \in J} m_i\alpha_i$ where $J \subseteq S$ and m_i's are positive integers. Since $\langle \sum_{\alpha_i \in J} m_i\alpha_i, \sum_{\alpha_i \in J} m_i\check{\alpha}_i\rangle \geq 0$ there exist an $\alpha_j \in J$ such that $\langle \sigma(\omega) - \tau(\omega), \check{\alpha}_j\rangle > 0$. Hence either $\langle \sigma(\omega), \check{\alpha}_j\rangle > 0$ or $\langle \tau(\omega), \check{\alpha}_j\rangle < 0$.
Case I: Let us assume $\langle \sigma(\omega), \check{\alpha}_j\rangle > 0$. Then $l(s_{\alpha_j}\sigma) = l(\sigma) + 1$ and $s_{\alpha_j}\sigma \in W^I$. Now

$ht(s_{\alpha_j}\sigma(\omega)-\tau(\omega))=m-1$. Hence, by induction $\tau=\phi_1 s_{\alpha_j}\sigma$ with $l(\tau)=l(\phi_1)+l(s_{\alpha_j}\sigma)$. Thus taking $\phi=\phi_1.s_{\alpha_j}$ we are done in this case.
<u>Case *II*</u>: Let us assume $\langle\tau(\omega),\check{\alpha_j}\rangle<0$. Then $l(s_{\alpha_j}\tau)=l(\tau)-1$ and $s_{\alpha_j}\tau\in W^I$. Since $\sigma(\omega)-s_{\alpha_j}\tau(\omega)=m-1$ by induction $s_{\alpha_j}\tau=\phi_2\sigma$ with $l(s_{\alpha_j}\tau)=l(\phi_2)+l(\sigma)$. Thus taking $\phi=s_{\alpha_j}\phi_2$ we are done in this case also. This completes the proof. $\square$

Corollary 1.9. *Let ω, w and I be as in lemma 1.6. Let $\sigma\in W^I$ be such that $\sigma(n\omega)\leq 0$ for some positive integer. Then, we have $w\leq\sigma$.*

Proof. The proof follows from lemma 1.6, 1.8 and the fact that ω is minuscule. $\square$

Corollary 1.10. *Let ω, w and I be as in lemma 1.6. Let $\sigma\in W^I$ be such that $\sigma(n\omega)\geq 0$ for some positive integer. Then, we have $\sigma\leq w$*

Proof. The proof follows from lemma 1.7, 1.8 and the fact that ω is minuscule. $\square$

2 Description of Schubert varieties in the Grassmannian having semi-stable points

In this section, we have the following notation. Let $G=GL_n(k)$ with characteristic of k is either zero or bigger than n. Let $r\in\{2,\cdots n-2\}$. Consider the action of a maximal torus T of $SL_n(k)$ on the Grassmannian $G_{r,n}$. Let B be a Borel subgroup of G containing T. Let $S=\{\alpha_1,\cdots\alpha_{n-1}\}$ be the set of simple roots with respect to B arranged in the ordering of the vertices in the Dynkin diagram of type A_{n-1}. Let $I_r=S\setminus\{\alpha_r\}$. We first note that $G_{r,n}$ is the homogeneous space $GL_n(k)/P_r$ where $P_r=BW_{I_r}B$ is the maximal parabolic subgroup of $GL_n(k)$ containing B associated to the simple root α_r. Let ω_r be the fundamental weight associated to the simple root α_r and let $\mathcal{L}_r$ denote the line bundle on $GL_n(k)/P_r$ corresponding to ω_r. We describe all Schubert cells in $GL_n(k)/P_r$ admitting semi-stable points for the above mentioned action of T with respect to the line bundle $\mathcal{L}_r$.

Some of the elementary facts about the combinatorics of W^{I_r} that is being used in this section can be found in [7]. For the convenience of the reader, we prove them here.

Lemma 2.1. *Let $w\in W^I, w\neq id$. Then there exists an $i\in\mathbb{N}$, $i\leq r$ and a sequence of positive integers $\{a_j\}$, $j=1,2,\ldots,r$ such that the following holds.*
(a) $a_j\geq j$ for all j, $i\leq j\leq r$
(b) $w=(s_{a_i}.s_{a_i-1}\ldots s_i)(s_{a_{i+1}}.s_{a_{i+1}-1}\ldots s_{i+1})\ldots(s_{a_r}.s_{a_r-1}\ldots s_r)$ with $l(w)=\sum_{j=i}^{r}(a_j-j+1)$

Proof. Let i be the least positive integer such that $s_{\alpha_i}\leq w$. The rest of the proof follows from braid relations in W. $\square$

Lemma 2.2. *Let $w,\tau\in W^I$. Write $w=(s_{a_i}.s_{a_i-1}\ldots s_i)(s_{a_{i+1}}.s_{a_{i+1}-1}\ldots s_{i+1})\ldots(s_{a_r}.s_{a_r-1}\ldots s_r)$ and $\tau=(s_{b_k}.s_{b_k-1}\ldots s_k)(s_{b_{k+1}}.s_{b_{k+1}-1}\ldots s_{k+1})\ldots(s_{b_r}.s_{b_r-1}\ldots s_r)$ be as in the lemma 2.1. Then $w\leq\tau\Leftrightarrow k\leq i$ and $b_j\geq a_j$ for all j, $i\leq j\leq r$.*

Proof. The proof follows from lemma 1.8 and the fact that $w(\omega_r) \geq \tau(\omega_r) \Leftrightarrow k \leq i$ and $b_j \geq a_j$ for all j, $i \leq j \leq r$. $\square$

Now, write $n = qr + t$ with $1 \leq t \leq r$ and let $\tau_r \in W^{I_r}$ be the unique element as in lemma 1.6 for the case when $\omega = \omega_r$. Then, τ_r must be of the form $\tau_r = (s_{a_1} \cdots s_1) \cdots (s_{a_r} \cdots s_r)$ where

$$a_i = \begin{cases} i(q+1) & if \quad i \leq t-1. \\ iq + (t-1) & if \quad t \leq i \leq r \end{cases}$$

Let $\tau^{n-r} \in W^{I_{n-r}}$ be the unique element as in lemma 1.7 for the case $\omega = \omega_r$. Then, we have $\tau_r = \tau^{n-r} w_0^{I_r}$ and $l(w_0^{I_r}) = l(\tau_r) + l(\tau^{n-r})$.

Let $w \in W^I$ be such that $w(n\omega_r) \leq 0$.

Then, we have

Lemma 2.3. *$\tau_r \leq w$ and $w\tau_r^{-1} \leq (\tau^{n-r})^{-1}$.*

Proof. Proof follows from corollary 1.8 and corollary 1.9. $\square$

For any such w, we describe the set $R^+(w^{-1})$.

Lemma 2.4. *$R^+(w^{-1})$ consists of roots of the form $\alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}$ for $1 \leq i \leq r$ where $j \neq a_k + 1$ for any $k < i$.*

Proof. We have $w^{-1} = (s_r \dots s_{a_r}) \dots (s_2 \dots s_{a_2}).(s_1 \dots s_{a_1})$, which is a reduced expression. Thus the elements of $R^+(w^{-1})$ are

$$\beta_{i,j-i+1} = (s_{a_1} \dots s_1).(s_{a_2} \dots s_2) \dots (s_{a_i} \dots s_{j+1}.\hat{s_j}.\hat{s_{j-1}} \dots \hat{s_i})(\alpha_j)$$

where $i \leq j \leq a_i$, $1 \leq i \leq r$, $\hat{}$ denotes omission of the symbols. We have,

$$(s_{a_i} \dots s_{j+1}.\hat{s_j}.\hat{s_{j-1}} \dots \hat{s_i})(\alpha_j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}$$

Since, $a_1 < a_2 < \cdots < a_r$, each $\beta_{i,j}$ is of the form

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}.$$

Now $j \neq a_k + 1$ for any $k < i$ follows from the fact that $l(w)$ is the same as the cardinality of $R^+(w^{-1})$. $\square$

Remark 2.5. *From the lemma it follows that the elements of $R^+(w^{-1})$ can be written in an array as follows:*

$$\begin{array}{llllllllllll}
\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,a_1} \\
\beta 2,1 & \beta_{2,2} & \cdots & \beta_{2,a_1} & \beta_{2,a_1+1} & \beta_{2,a_1+2} & \cdots & \beta_{2,a_2-1} \\
\beta 3,1 & \beta_{3,2} & \cdots & \beta_{3,a_1} & \beta_{3,a_1+1} & \beta_{3,a_1+2} & \cdots & \beta_{3,a_2-1} & \beta_{3,a_2} & \cdots & \beta_{3,a_3-2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\beta r,1 & \beta_{r,2} & \cdots & \beta_{r,a_1} & \beta_{r,a_1+1} & \beta_{r,a_1+2} & \cdots & \beta_{r,a_2-1} & \beta_{r,a_2} & \cdots & \beta_{r,a_3-2} & \cdots & \beta_{r,a_r-r+1}
\end{array}$$

where the array has r rows, and the length of the i-th row is $a_i-(i-1)$. Note that $\beta_{1,a_1}=\alpha_{a_1}$, and for $2\le i\le r$, $\beta_{i,a_i-i+1}=\alpha_{a_i}$, only if $a_i\ge a_{i-1}+2$. In this case, for all j, $i\le j\le r$, $\beta_{j,a_{i-1}-i+2}=\beta_{i-1,a_{i-1}-i+2}+\alpha_{a_{i-1}+1}+\alpha_{a_{i-1}+2}+\cdots+\alpha_{a_j}$ and $\beta_{j,a_{i-1}-i+3}=\alpha_{a_{i-1}+2}+\alpha_{a_{i-1}+3}+\cdots+\alpha_{a_j}$. If $a_i=a_{i-1}+1$, then $a_i-i+1=a_{i-1}-(i-1)+1$, therefore, the $(i-1)$-th and i-th rows have same length. In this case for all j, $i\le j\le r$, $\beta_{j,a_i-i+1}=\beta_{i-1,a_i-i+1}+\alpha_{a_{i-1}+1}+\alpha_{a_{i-1}+2}+\cdots+\alpha_{a_j}$.

For any $w\in W^I$, let $X(w):=\overline{BwP_r/P_r}$ denote the Schubert variety in $GL_n(k)/P_r$.

We recall $BwP_r/P_r=U_wwP_r$, where U_w is the product $\prod_{\alpha\in R^+(w^{-1})}U_\alpha$ of the root groups U_α, and we describe below the ordering of roots in which the product is taken.

Consider the open set

$$V:=\{\prod_{\beta_{ij}\in R^+(w^{-1})}u_{\beta_{ij}}(x_{\beta_{ij}}).w.P_r : x_{\beta_{ij}}\neq 0,\ \forall\beta_{ij}\in R^+(w^{-1})\}$$

of $X(w)$ in GL_n/P_r where the order in which the product is taken is as follows: Put a partial order on $R^+(w^{-1})$ by declaring $\beta_{ij}\le\beta_{kl}$ if either $i=k$ and $j\ge l$ or if $i<k$. Now we take the product so that whenever $\beta_{ij}\le\beta_{kl}$, $u_{\beta_{ij}}(x_{\beta_{ij}})$ appears on the right hand side of $u_{\beta_{kl}}(x_{\beta_{kl}})$. Note that $u_{\beta_{ij}}(x_{\beta_{ij}})$'s commute with each other, since $\beta_{i_1,j_1},\beta_{i_2,j_2}\in R^+(w^{-1})$ implies $\beta_{i_1,j_1}+\beta_{i_2,j_2}$ is not a root. This follows from the fact that no element of $R^+(w^{-1})$ starts or ends with α_{a_k+1}, for any k, $1\le k\le r-1$ (i.e. for all $\beta_{i,j}\in R^+(w^{-1})$ and $1\le k\le r-1$, $\beta_{i,j}-\alpha_{a_k+1}\neq 0$ is not a root.)

Now the natural action of the maximal torus T on $GL_n(k)/P_r$, induces an action of T on V .

Lemma 2.6. *Consider the torus $T'=\prod_{\beta\in R^+(w^{-1})}G_{m,\beta}$ where $G_{m,\beta}=G_m$ for each $\beta\in R^+(w^{-1})$. We have a natural action of T on T' through the homomorphism of algebraic groups $\Phi:T\to T'$ defined by $\Phi(t)=(\beta(t))_\beta$ for all $t\in T$. The map $V\to T'$ defined by $\prod u_\beta(x_\beta)w.P\mapsto(x_\beta)_\beta$ is a T-equivariant isomorphism of varieties.*

Proof. Proof is easy. □

We now describe all the Schubert varieties admitting semi-stable points.
Let $n=qr+t$, with $1\le t\le r$ and let $w\in W^{I_r}$.

Lemma 2.7. *Then the following are equivalent:*
(1) $X(w)^{ss}_T(L_r)$ is non-empty.
(2) $\tau_r\le w$ and $w\tau_r^{-1}\le(\tau^{n-r})^{-1}$.
(3) $w=(s_{a_1}\cdots s_1)\cdots(s_{a_r}\cdots s_r)$, where $\{a_i : i=1,2\cdots r\}$ is an increasing sequence of positive integers such that $a_i\ge i(q+1)\ \forall\ i\le t-1$ and $a_i=iq+(t+1)\ \forall\ t\le i\le r$.

Proof. By Hilbert-Mumford criterion (theorem 2.1 of [3]) a point $x\in G/P_r$ is semi-stable if and only if $\mu^L(\sigma x,\lambda)\le 0$ for all $\lambda\in\overline{C(B)}$ and for all $\sigma\in W$. By the lemma 2.1 of

[6], this statement is equivalent to $\langle -w_\sigma(\omega), \lambda\rangle \geq 0$ for all $\lambda \in \overline{C(B)}$ and for all $\sigma \in W$, where $w_\sigma \in W^{I_r}$ is such that $\sigma x \in U_{w_\sigma} w_\sigma P_r$. Thus, by corollary 1.8 applied to the situation $\omega = \omega_r$, a point x is semi-stable if and only if x is not in the W- translates of $U_\tau \tau P_r$ with $\tau \in W^{I_r}$ and $\tau_r \not\leq \tau$.

Now, for a $w \in W^{I_r}$, $X(w)$ is not contained in the finite union $\bigcup_{\tau \not\geq \tau_r} U_\tau \tau P_r$ if and only if $\tau_r \leq w$. The second condition $w\tau_r^{-1} \leq (\tau^{n-r})^{-1}$ is an immediate consequence when $w \geq \tau_r$. This completes the proof. $\square$

Proposition 2.8. *Let $X_{i,j}$ denote the regular function on V defined by $\prod u_{\beta_{kl}}(x_{\beta_{kl}})w.P \mapsto x_{\beta_{ij}}$ for all $1 \leq i \leq r-1$ and $1 \leq j \leq a_i - i + 1$; and let $Y_{i,j} := \frac{X_{i,a_i-i+1}.X_{i+1,j}}{X_{i,j}.X_{i+1,a_i-i+1}}$. Then the ring of T-invariant regular functions is generated by $Y_{i,j}, Y_{i,j}^{-1}$, where $1 \leq j \leq a_i - i$, for each i, and $1 \leq i \leq r-1$; $Y_{i,j}$ are algebraically independent.*

Proof. Now, consider the homomorphism of tori,

$$T \xrightarrow{\Psi} T' \ \ defined \ \ by$$

$$\Psi(t) = (t^{\beta_{ij}}), \ \ i = 1, 2 \cdots r, j = 1, 2 \cdots a_i - i + 1.$$

Proof of the proposition follows from the following claim.
Claim: $E_{i,a_i-(i-1)} - E_{i+1,a_i-(i-1)} - E_{i,j} + E_{i+1,j}; i = 1, 2 \cdots r-1$ *and* $j = 1, 2 \cdots a_i - i$ forms a basis for $Ker(\Psi^* : X(T) \longrightarrow X(T'))$, where $E_{i,k}$ is the matrix with 1 in the $(i,k)^{th}$ place and 0 elsewhere.
Proof of the claim: Now any character of T' is of the form $(t_\beta) \mapsto \prod t_\beta^{m_\beta}$ where m_β are integers. Now such a character is T-invariant iff the sum $\sum_\beta m_\beta \beta$ is zero. Plugging in the expression of β's in terms of the simple roots α_k's and noting that they are linearly independent we get a set of linear equations over $\mathbb{Z}$, by equating to zero the coefficient of each α_k. Let us denote by $R(p)$, $1 \leq p \leq r$ the set of roots appearing in p-th row of the array described above; and let $C(q)$, $1 \leq q \leq a_r - (r-1)$ denote the set of roots appearing in the q-th column of the array.

Comparing the coefficient of α_1, we have $\sum_{\beta \in C(1)} m_\beta = 0$.
Comparing the coefficient of α_2, and using the above observation, we get $\sum_{\beta \in C(2)} m_\beta = 0$.
Proceeding this way, we get

$$\sum_{\beta \in C(j)} m_\beta = 0 \ \ \forall j, \ 1 \leq j \leq a_1.$$

Let k be the least positive integer such that $\alpha_k + \cdots + \alpha_{a_i}$ is the first root in the column $C(a_1 + 1)$.
Comparing the coefficient of α_k, we get $\sum_{\beta \in C(a_1+1)} m_\beta = 0$.
Proceeding this way, we get

$$\sum_{\beta \in C(j)} m_\beta = 0 \ \ \forall j, \ 1 \leq j \leq a_r - r + 1.$$

Now comparing the coefficient of α_{a_r}, we get $\sum_{\beta\in R(r)} m_\beta = 0$.
Comparing the coefficient of $\{\alpha_j : j = a_{r-1}, 2 + a_{r-1}, \cdots a_r\}$, we get

$$\sum_{\beta\in R(r-1)} m_\beta + \sum_{\beta\in R(r)} m_\beta = 0.$$

Thus we have

$$\sum_{\beta\in R(r-1)} m_\beta = 0.$$

Proceeding this way, we get

$$\sum_{\beta\in R(i)} m_\beta = 0 \ \forall i,\ 1 \leq i \leq r.$$

□

3 Description of the action of the Weyl group on the quotient ${}_{T}\backslash\backslash G^{ss}_{r,n}(\mathcal{L}_r)$

In this section, we describe the action of the Weyl group on the quotient ${}_{T}\backslash\backslash G^{ss}_{r,n}(\mathcal{L}_r)$.

We first write down the stabiliser of $X(w)$ in W. Let $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W^{I_r}$ be such that $w \geq \tau_r$. Then, we have

Lemma 3.1. *Description of the set $\{s_i : s_i(X(w)) \subseteq X(w), i = 1, 2, \cdots n-1\}$:*
1. $\{s_j : 1 \leq j \leq a_1 - 2\}$.
2. $\{s_j : a_p + 2 \leq j \leq a_{p+1} - 2, p = 1, 2, \cdots r-1\}$.
3. $\{s_{a_p - 1} : p = 1, 2, \cdots r\}$.
4. $\{s_{a_p} : p = 1, 2, \cdots r\}$.

Proof. Proof uses braid relations of the Weyl group S_n. □

We now explicitely describe the action of the stabilisers on

Proposition 3.2. *Description of the action:*
1. *s_j interchanges $Y_{i,j}$ and $Y_{i,j+1}$ for $i = 1, 2, \cdots r-1$, and keeps all other $Y_{i,k}$'s fixed.*
2. *s_j interchanges $Y_{i,j-p}$ and $Y_{i,j-p+1}$ for $p+1 \leq i \leq r-1$, and keeps all other $Y_{i,k}$'s fixed.*
3(a). *If $2 \leq p \leq r$, then s_{a_p-1} fixes all the $Y_{i,k}, 1 \leq i \leq p-1$.*
(b). *If $p \leq i \leq r-1$, $a_p - p = a_i - i$ and $1 \leq k \leq a_p - p$, then $s_{a_p-1}(Y_{i,a_p-p}) = Y^{-1}_{i,a_p-p}$, and $s_{a_p-1}(Y_{i,k}) = Y_{i,k}.Y^{-1}_{i,a_p-p}$.*
(c). *If $p+1 \leq i \leq r-1$, $a_i - i \gneq a_p - p$, then $s_{a_p-1}(Y_{i,a_p-p}) = Y_{i,a_p-p+1}$, and keeps all other*

$Y_{i,k}$'s fixed.

$4(a)$. *$2 \leq p \leq r-1$, and $a_p = a_{p-1}+1$.*

(i). *If $3 \leq p \leq r$ and $1 \leq k \leq a_{a_{p-2}-p+2}$, then $s_{a_p}(Y_{p-2,k}) = Y_{p-2,k}.Y_{p-1,k}.Y^{-1}_{p-1,a_{p-2}-p+3}$.*

(ii). *If $1 \leq k \leq a_p - p$ then $s_{a_p}(Y_{p-1,k}) = Y^{-1}_{p-1,k}$ and $s_{a_p}(Y_{p,k}) = Y_{p,k}.Y_{p-1,k}$.*

(iii). *$Y_{i,k}$'s are fixed for $i \neq p-2, p-1, p$ and $1 \leq k \leq a_i - i$.*

$(b)(i)$. *If $1 \leq i \leq p-1$ or $a_p - p + 1 \leq k \leq a_r$, $Y_{i,k}$'s are fixed.*

(ii). *If $i = p$ and $1 \leq k \leq a_p - p$ then $s_{a_p}(Y_{p,k}) = 1 - Y_{p,k}$.*

(iii) *If $p+1 \leq i \leq r-1$ and $1 \leq k \leq a_p - p$, then, $s_{a_p}(Y_{i,k}) = \frac{1-\prod_{m=p}^{i}(Y_{m,k}/Y_{m,a_p-p+1})}{1-\prod_{m=p}^{i-1}(Y_{m,k}/Y_{m,a_p-p+1})} \times Y_{i,a_p-p+1}$.*

(c). *Action of s_{a_r}:*

(i). *If $a_r = a_{r-1}+1$ then $s_{a_r}(Y_{r-2,k}) = Y_{r-2,k}.Y_{r-1,k}.Y^{-1}_{r-1,a_{r-2}-r+3}$, for $1 \leq k \leq a_{r-2} - r + 2$ and $s_{a_r}(Y_{r-1,k}) = Y^{-1}_{r-1,k}$, for $1 \leq k \leq a_r - r$.*

(ii). *If $a_{r-1}+2 \leq a_r$ then $Y_{r,k}$'s are fixed for $1 \leq k \leq a_r - r + 1$.*

Proof. Proof is essntially based on the following properties of groups with BN-pair and commutator relations:

$$(i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 0 & \frac{-1}{x} \end{pmatrix}, \text{ and}$$

$$(ii)\ [u_\alpha(x_\alpha), u_\beta(x_\beta)] = \begin{cases} u_{\alpha+\beta}(x_\alpha.x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_j - \epsilon_k\ , i < j < k; \\ u_{\alpha+\beta}(-x_\alpha.x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_k - \epsilon_i\ , k < i < j. \end{cases}$$

We first consider the action of W on the $X_{j,k}$'s and then describe resulting action on the $Y_{j,k}$'s. If $1 \leq i \leq a_1 - 2$ then s_i interchanges $X_{j,i}$ and $X_{j,i+1}$ for all j, $1 \leq j \leq r$. Therefore, it follows that s_i interchanges $Y_{j,i}$ and $Y_{j,i+1}$ for all j, $1 \leq j \leq r-1$ and keeps all other $Y_{j,k}$'s fixed. Similarly for $p \geq 2$ and $a_p + 2 \leq a_{p+1}$, if $a_p + 2 \leq i \leq a_{p+1} - 2$, s_i interchanges $X_{j,i-p}$ and $X_{j,i-p+1}$. Thus s_i interchanges $Y_{j,i-p}$ and $Y_{j,i-p+1}$ for all j, $i+1 \leq j \leq r-1$ and keeps all other $Y_{j,k}$s fixed. Now, we compute the actions of s_{a_i-1}, s_{a_i} and s_{a_i+1}.

Action of s_{a_i+1} for each i, $1 \leq i \leq r-1$

Case I: $a_i + 2 \leq a_{i+1}$ In this case we have

$$\begin{aligned} & s_{a_i+1}w \\ & = s_{a_1+1}.(s_{a_1} \dots s_1).(s_{a_2} \dots s_2) \dots (s_{a_r} \dots s_r) \\ & = (s_{a_1} \dots s_1) \dots (s_{a_i+1}.s_{a_i} \dots s_i) \dots (s_{a_r} \dots s_r) \end{aligned}$$

which is a reduced expression and $s_{a_i+1}.w \in W^I$ by lemma 1.12. Now lemma 1.13 implies that $s_{a_i+1}.w \geq w$. Hence, $X(w)$ is not stable under the action of s_{a_i+1}.

Case II: $a_i + 1 = a_{i+1}$ In this case $s_{a_i+1} = s_{a_{i+1}}$ and the action will be described in the later part of this paragraph. In fact we see that in this case $(s_{a_i+1}w)^I = w$. Hence $X(w)$ is stable under the action of s_{a_i+1}.

Action of s_{a_i-1}

In case $i = 1$, we may assume that $a_1 \neq 1$, and for $i \geq 2$, $a_{i-1} \neq a_i - 1$. Now s_{a_i-1} interchanges the $(a_i - i)$-th and $(a_i - i + 1)$-th columns of each of the j-th row, of the array

of roots $R^+(w^{-1})$, for $i \leq j \leq r$; thus s_{a_i-1} interchanges X_{j,a_i-i} and X_{j,a_i-i+1} for each j, $i \leq j \leq r$. Therefore, the action of s_{a_i-1} is as follows:
(1) s_{a_i-1} fixes all the $Y_{j,k}$, for $1 \leq j \leq i-1$, for $i \geq 2$.
(2) For $j \geq i \leq r-1$ and $a_i - i = a_j - j$, $Y_{j,a_i-i} \mapsto Y_{j,a_i-i}^{-1}$, and for $Y_{j,k} \mapsto Y_{j,k}.Y_{j,a_i-i}^{-1}$ for $1 \leq k < a_i - i$.
(3) For $i+1 \leq j \leq r-1$ if $a_j - j > a_i - i$, then s_{a_i-1} interchanges Y_{j,a_i-i} and Y_{j,a_i-i+1} and keeps all other $Y_{j,k}$'s fixed.

Action of s_{a_i} for $1 \leq i \leq r$

Let us show that $X(w)$ is stable under the action of each of the s_{a_i}. Let

$$w = (s_{a_1} \dots s_1).(s_{a_2} \dots s_2) \dots (s_{a_r} \dots s_r)$$

Thus

$$s_{a_i} w = (s_{a_1} \dots s_1) \dots (s_{a_{i-2}} \dots s_{i-2}).s_{a_i}.(s_{a_{i-1}} \dots s_{i-1}).(s_{a_i} \dots s_i) \dots (s_{a_r} \dots s_r)$$

Case 1: $i = 1$, or $a_{i-1} + 2 \leq a_i$ for $i \geq 2$. In this case it is clear that

$$s_{a_i} w = (s_{a_1} \dots s_1) \dots (s_{a_{i-2}} \dots s_{i-2}).(s_{a_{i-1}} \dots s_{i-1}).(s_{a_i-1} \dots s_i) \dots (s_{a_r} \dots s_r)$$

which, by lemma 1.12 and 1.13, is in W^I and $s_{a_i} w \leq w$.
Case 2: $a_{i-1} + 1 = a_i$. Note that,

$$w_1 = (s_{a_{i-1}} \dots s_{i-1}).(s_{a_i} \dots s_i) \in W^J$$

where $J = S \setminus \{\alpha_i\}$. Now,

$$\begin{array}{rcl} w_1(\omega_i) & = & \omega_i - \sum_{j=i-1}^{a_{i-1}} \alpha_j - \sum_{j=i}^{a_i} \alpha_j \\ \Rightarrow s_{a_i} w_1(\omega_i) & = & s_{a_i}(\omega_i) - \sum_{j=i-1}^{a_{i-1}} \alpha_j - \sum_{j=i}^{a_i} \alpha_j \end{array}$$

Now, if $a_i = i$, then $a_{i-1} = i-1$; so $s_{a_i} w_1 = s_i.s_{i-1}.s_i = s_{i-1}.s_i.s_{i-1} = w_1.s_{i-1}$. Otherwise, $a_i \neq i$. This implies that $s_{a_i}(\omega_i) = \omega_i$. Therefore, $s_{a_i} w_1(\omega_i) = w_1(\omega_i)$. Hence, by lemma 1.3, we get $s_{a_i} w_1 = w_1.s_\alpha$ for some $\alpha \in J$. This gives $w_1^{-1} s_{a_i} w_1 = s_{w_1^{-1}(\alpha_{a_i})} = s_\alpha$. Now it follows that $w_1^{-1}(\alpha_{a_i}) = \alpha_{i-1}$. Hence, $s_{a_i} w_1 = w_1.s_{i-1}$. Therefore, in both the sub-cases $s_{a_i}.w = w.s_{i-1}$; in particular $(s_{a_i}.w)^I = w$. Now we shall compute the action of s_{a_i}, for $1 \leq i \leq r$.

Case I: $2 \leq i \leq r-1$ *and* $a_i = a_{i-1} + 1$. In this case, s_{a_i} interchanges $X_{i,k}$ and $X_{i-1,k}$ for $1 \leq k \leq a_i - i + 1$ and keeps all other $X_{j,k}$'s fixed. Hence, the action of s_{a_i} on the $Y_{j,k}$'s is as follows:
(1) If $i \geq 3$, $Y_{i-2,k} \mapsto Y_{i-2,k}.Y_{i-1,k}.Y_{i-1,a_{i-2}-i+3}^{-1}$ for $1 \leq k \leq a_{i-2} - i + 2$
(2) $Y_{i-1,k} \mapsto Y_{i-1.k}^{-1}$ for $1 \leq k \leq a_i - i$.
(3) $Y_{i,k} \mapsto Y_{i,k}.Y_{i-1,k}$ for $1 \leq k \leq a_i - i$.
(4) $Y_{j,k}$ is fixed for $1 \leq k \leq a_j - j$ for each $j \neq i-2, i-1, i$.

Case II: $a_i \geq a_{i-1}+2$ for $2 \leq i \leq r-1$, or $i=1$. In this case s_{a_i} changes only the i-th row and the (a_i-i+1)-th column of the array of roots $R^+(w^{-1})$. The resulting i-th row turns out to be

$\alpha_1+\alpha_2+\cdots+\alpha_{a_i-1},\ \alpha_2+\cdots+\alpha_{a_i-1},\ \cdots,\alpha_{a_1}+\cdots+\alpha_{a_i-1},\ \alpha_{a_1+2}+\cdots+\alpha_{a_i-1},\cdots,\alpha_{a_i-1},\ -\alpha_{a_i}$

and the transpose of the (a_i-i+1)-th column turns out to be

$$-\alpha_{a_i},\ \alpha_{a_i+1}+\cdots+\alpha_{a_{i+1}},\ \alpha_{a_i+1}+\cdots+\alpha_{a_{i+2}},\ \cdots,\alpha_{a_i+1}+\cdots+\alpha_{a_r}$$

Let $\beta_{j,k}$ be any root which is fixed under the action of s_{a_i}. and let $\beta_{p,q}$ be any root of the i-th row or the (a_i-i+1)-th column, i.e. either $p=i$ or $q=a_i-i+1$. We claim that $u_{\beta_{i,j}}(X_{i,j})$ and $u_{s_{a_i}\beta_{p,q}}(X_{p,q})$ commute. This follows from the fact that $\beta_{j,k}-\alpha_{a_i}\notin R^+(w^{-1})$ and the observation that for any root $\beta\in R^+(w^{-1})$ and $1\leq m\leq r$ $\beta-\alpha_{a_m+1}\notin R^+$. Let us denote by M the sub-array consisting of $\beta_{k,l}$ where $k\geq i$ and $1\leq l\leq a_i-i+1$. Then

$$\begin{aligned}&s_{a_i}.(u_{\beta_{r,1}}(X_{r,1}).u_{\beta_{r,2}}(X_{r,2})\ldots u_{\beta_{r,a_r-r+1}}(X_{r,a_r-r+1}).u_{\beta_{r-1,1}}(X_{r-1,1}).u_{\beta_{r-1,2}}(X_{r-1,2})\\&\ldots u_{\beta_{r-1,a_{r-1}-r+2}}(X_{r-1,a_{r-1}-r+2})\ldots u_{\beta_{1,1}}(X_{1,1}).u_{\beta_{1,2}}(X_{1,2})\ldots u_{\beta_{1,a_1}}(X_{1,a_1})).w.P\\&=(\textstyle\prod_{\beta_{k,l}\notin M}u_{\beta_{k,l}}(X_{k,l})).s_{a_i}.(u_{\beta_{r,1}}(X_{r,1}).u_{\beta_{r,2}}(X_{r,2})\ldots u_{\beta_{r,a_i-i+1}}(X_{r,a_i-i+1})).(u_{\beta_{r-1,1}}(X_{r-1,1})\cdots\\&u_{\beta_{r-1,2}}(X_{r-1,2})\ldots u_{\beta_{r-1,a_i-i+1}}(X_{r-1,a_i-i+1}))\ldots u_{\beta_{i,1}}(X_{i,1}).u_{\beta_{i,2}}(X_{i,2})\ldots u_{\beta_{i,a_1-i+1}}(X_{i,a_i-i+1})).w.P\end{aligned}$$

Thus the action of s_{a_i}, in this case is as follows:

$$X_{i,a_i-i+1}\mapsto X^{-1}_{i,a_i-i+1};\quad X_{i,k}\mapsto X_{i,k}.X^{-1}_{i,a_i-i+1}\text{ for }k\leq a_i-i$$

$$X_{j,k}\mapsto X_{j,k}-\frac{X_{j,a_i-i+1}.X_{i,k}}{X_{i,a_i-i+1}}\quad\text{for }i+1\leq j\leq r\text{ and }1\leq k\leq a_i-i$$

$$X_{j,a_i-i+1}\mapsto -X_{j,a_i-i+1}/X_{i,a_i-i+1}\text{ for }i+1\leq j\leq r$$

From this the resulting action on the $Y_{j,k}$ turns out to be as follows:
(1) s_{a_i} fixes $Y_{j,k}$'s provided $j\leq i-1$ or $k\geq a_i-i+1$.
We now make the convention that $Y_{j,k}:=1$ if $k\geq a_j-j+1$ or if $j\geq r$.
(2) $j=i$. Here, for $k\leq a_i-i$,

$$\begin{aligned}Y_{i,k}&=\tfrac{X_{i,a_i-i+1}.X_{i+1,k}}{X_{i+1,a_i-i+1}.X_{i,k}}\\ \therefore\quad s_{a_i}(Y_{i,k})&=\frac{X^{-1}_{i,a_i-i+1}.(X_{i+1,k}-\frac{X_{i+1,a_i-i+1}.X_{i,k}}{X_{i,a_i-i+1}})}{X_{i,k}.X^{-1}_{i,a_i-i+1}.(-X_{i+1,a_i-i+1}/X_{i,a_i-i+1})}\\&=1-Y_{i,k}\end{aligned}$$

(3) $i+1\leq j\leq r-1$ and $1\leq k\leq a_i-i$. Define $Y'_{j,k}=(X_{i,a_i-i+1}.X_{j,k})/(X_{j,a_i-i+1}.X_{i,k})$. Then, we have $s_{a_i}(Y_{j,k})=1-Y_{j,k}$. It follows that $Y_{j,k}=Y'_{j+1,k}.{Y'_{j,k}}^{-1}.Y_{j,a_i-i+1}$. Hence, $s_{a_i}(Y_{j,k})=\frac{1-Y'_{j+1,k}}{1-Y'_{j,k}}.Y_{j,a_i-i+1}$. Now,

$$\begin{aligned}Y'_{j,k}&=\textstyle\prod_{m=i}^{j-1}\frac{X_{m,a_i-i+1}.X_{m+1,k}}{X_{m+1,a_i-i+1}.X_{m,k}}\\&=\textstyle\prod_{m=i}^{j-1}\{(\frac{X_{m,a_m-m+1}.X_{m+1,k}}{X_{m+1,a_m-m+1}.X_{m,k}})\times(\frac{X_{m,a_m-m+1}.X_{m+1,a_i-i+1}}{X_{m+1,a_m-m+1}.X_{m,a_i-i+1}})^{-1}\}\\&=\textstyle\prod_{m=i}^{j-1}(Y_{m,k}/Y_{m,a_i-i+1})\end{aligned}$$

Thus we have,

$$\boxed{s_{a_i}(Y_{j,k}) = \frac{1-\prod_{m=i}^{j}(Y_{m,k}/Y_{m,a_i-i+1})}{1-\prod_{m=i}^{j-1}(Y_{m,k}/Y_{m,a_i-i+1})} \times Y_{j,a_i-i+1}}$$

Case III: Action of s_{a_r}: (1) If $a_r = a_{r-1}+1$, then s_{a_r} interchanges $X_{r-1,k}$ and $X_{r,k}$, $1 \leq k \leq a_r - r + 1$. A straightforward checking proves as in *Case I* above, that in this case the action of s_{a_r} is as follows:

$$\begin{array}{ll} Y_{r-2,k} \mapsto Y_{r-2,k}.Y_{r-1,k}.Y^{-1}_{r-1,a_{r-2}-r+3} & \text{for } 1 \leq k \leq a_{r-2} - r + 2 \\ Y_{r-1,k} \mapsto Y^{-1}_{r-1.k} & \text{for } 1 \leq k \leq a_r - r \end{array}$$

(2) If $a_r \geq a_{r-1}+2$, s_{a_r} changes only $X_{r,k}$'s for $1 \leq k \leq a_r - r + 1$, as follows:

$$\begin{array}{ll} X_{r,k} \mapsto X_{r,k}.X^{-1}_{r,a_r-r+1} & \text{for } 1 \leq k \leq a_r - r \\ X_{r,a_r-r+1} \mapsto X^{-1}_{r,a_r-r+1} & \end{array}$$

It can be easily checked from here that the $Y_{i,j}$'s are all fixed by s_{a_r}. □

4 A stratification of ${}_N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$.

In this section, we give a stratification of ${}_N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$.

Lemma 4.1. *Let $w \in W^{I_2}$. Let $x \in U_w w {P_2}^{ss}$ be such that x is not in the W -translate of $X_\tau, \tau < w$. If $\sigma(x) \in U_w w P_2$, then $\sigma \in$ Stabiliser of $X(w)$ in W.*

Proof. Let $\sigma \in W$ be of minimal length such that $\sigma x \in U_w w P_2$. Then $\sigma = \sigma_1.\sigma_2$ with $l(\sigma) = l(\sigma_1) + l(\sigma_2)$ and $\sigma_2.w \in W^I, w \leq \sigma_2 w$.
Let σ_2 be of maximal length with this property. So $\sigma.w = s_{m+t+1}s_{m+t}\cdots s_{m+1}w, t \geq 1$, and $w = (s_m \cdots s_1)(s_{n-1}\cdots s_2)$.
Now, $\sigma_1(\sigma_2 u_w w P_2) \in U_w w P_2$. . . . (1)
Since σ_2 is of maximal length $s_{m+j} \not\leq \sigma_1$ for some $j \geq 1$. . . . (2)
Now, $\sigma_2 x \in U_{\sigma_2 w}\sigma_2 w P_2$. Since $l(\sigma) = l(\sigma_1) + l(\sigma_2)$ and $\sigma^{-1}(\alpha_{m+t+1}) < 0$, σ_2 is of maximal length, we may assume that $\sigma_1(\alpha_j) > 0$. . . . (3)
From $(1), (2)$ and (3), σ_1 must take a reduced form as

$$\begin{aligned} \sigma_1 &= (\phi s_{m+t-1}s_{m+t+1}s_{m+t})\sigma_2 \\ &= \phi s_{m+t-1}(s_{m+t+1}s_{m+t}s_{m+t+1})s_{m+t}\sigma_2' \\ &= \phi s_{m+t-1}s_{m+t}s_{m+t+1}\sigma_2' \end{aligned}$$

This contradicts the assumption that $l(\sigma) = l(\sigma_1) + l(\sigma_2)$.
This completes the proof. □

The longest element of W^{I_2} is

$$w_0^I = (s_{n-2}.s_{n-3}\ldots s_1).(s_{n-1}.s_{n-2}\ldots s_2)$$

and the unique minimal element τ_2 of W^I such that $\tau_2(n\omega_2) \leq 0$ is

$$\tau_2 = (s_{\lceil \frac{n-1}{2} \rceil}.s_{\lceil \frac{n-1}{2} \rceil -1} \dots s_1).(s_{n-1}.s_{n-2} \dots s_2)$$

Therefore any element $w \in W^I$ such that $X(w)^{ss}_T(\mathcal{L}_2) \neq$ is of the form

$$w = (s_m.s_{m-1} \dots s_{\lceil \frac{n-1}{2} \rceil}.s_{\lceil \frac{n-1}{2} \rceil -1} \dots s_1).(s_{n-1}.s_{n-2} \dots s_2)$$

with $m \geq \lceil \frac{n-1}{2} \rceil$.

Proposition 4.2. *Let $r = 2, w = (s_m...s_1)(s_{n-1}...s_2)$, $\lceil \frac{n-1}{2} \rceil \leq m \leq n-2$. We can arrange the Y_{ij}'s as $Y_1, Y_2, \cdots Y_{m-1}$ with*

$$s_i(Y_i) = Y_{i+1},$$

$$s_i(Y_j) = Y_j \ \ if \ \ j = k, i+1 \ \ and \ \ i = 1, 2 \cdots m-2,$$

$$s_{m-1}(Y_i) = Y_i.Y_{m-1}^{-1}, \ \ if \ \ i \leq m-2,$$

$$s_{m-1}(Y_{m-1}) = Y_{m-1}^{-1},$$

$$s_m(Y_i) = 1 - Y_i \ \ for \ \ i = 1, 2, \cdots m-1.$$

Further, we have

$$s_i(Y_j) = Y_j \ \forall \ i = m+2, \cdots n-1, \ when \ m \leq n-3$$

and

$$s_{n-1}(Y_j) = Y_j^{-1} \ \forall \ j \ when \ m = n-2.$$

Proof. Proof follows from the proposition 3.2. □

Let w be as in the proposition 4.2. Now, let T_{m-1} be a maximal torus of $\mathbb{P}GL_m$, R_m is the root system of $\mathbb{P}GL_m$. Here, the Weyl group is S_m, the symmetric group on m symbols. Let $U = \{t \in T : e^{\alpha}(t) \neq 1, \ \ \alpha \in R_m\}$. Clearly, U is S_m-stable. On the other hand, S_m stabilises $(U_w w P_2/P_2)^{ss}_T(\mathcal{L}_2)$. Let $Y(w) = {}_T\backslash\backslash(U_w w P_2)^{ss}_T(\mathcal{L}_2)$. Then, we have

Corollary 4.3. *There is a S_m-equivariant isomorphism $\Psi_1 : Y(w) \xrightarrow{\sim} U$ such that $\Psi_1^*(e^{\alpha_i + \cdots + \alpha_{m-1}}) = Y_i, \ \ 1 \leq i \leq m-1$.*

Proof. Proof follows from proposition 4.2. □

Let $\mathfrak{h}_m$ be a Cartan subalgebra of $\mathfrak{sl}_{m+1}$, $\mathbb{P}(\mathfrak{h}_m)$ be the projective space and $R_m \subseteq \mathfrak{h}_m^*$ be the root system. Let V_m be the open subset of $\mathbb{P}(\mathfrak{h}_m)$ defined by

$$V_m := \{x \in \mathbb{P}(\mathfrak{h}_m) : \alpha(x) \neq 0, \forall \alpha \in R_m\}.$$

Clearly V_m is S_{m+1}-stable.

Corollary 4.4. *Let* $w = (s_m...s_1)(s_{n-1}...s_2)$, $\lceil \frac{n-1}{2} \rceil \leq m \leq n-2$. *Then, there is a* S_{m+1}*-equivariant isomorphism* $\Psi_2 : Y(w) \xrightarrow{\sim} V$ *of affine varieties.*

Proof. For $i = 1, 2 \cdots m-1$, take $Z_i = \frac{\alpha_i + \cdots + \alpha_m}{\alpha_m}$ and define Ψ_2 such that $\Psi_2^*(Z_i) = Y_i$. □

With notations as above and taking $t = [\frac{n-1}{2}]$ and $m = \lceil \frac{n-1}{2} \rceil$ we have

Theorem: ${}_N\backslash\backslash G_{2,n}^{ss}(\mathcal{L}_2)$ has a stratification $\bigcup_{i=0}^{t} C_i$ where $C_0 = {}_{S_{m+1}}\backslash \mathbb{P}(\mathfrak{h}_m)$, and $C_i = {}_{S_{i+m+1}}\backslash V_{i+m}$.

Proof. Proof follows from lemma 4.1, proposition 4.2 and corollary 4.4. □

5 Flag variety as a GIT quotient of flag variety of higher dimension

Let $G = GL_{n+1}(k)$. Let T be a maximal torus of $SL_{n+1}(k)$. Let B_{n+1} be a Borel subgroup of G containing T. Let $S = \{\alpha_i : i = 1, 2, \cdots n\}$ denote the set of simple roots with respect to B_{n+1}, let $W = S_{n+1}$ be the Weyl group. Let s_i be the simple reflection corresponding to the simple root α_i. Let $I := S \setminus \{\alpha_n\}$, let W_I be subgroup of W generated by $\{s_i : i \in I\}$ and $w_{0,I}$ denote the longest element of W_I.

Lemma 5.1. *Let* $\chi = \sum_{i=1}^{n} m_i \alpha_i$ *be a regular dominant character, where* $m_i \in \mathbb{N}$, $m_{i+1} > m_i$ *for* $1 \leq i \leq n-1$. *Let* $w \in W$. *Then* $w(\chi) \leq 0 \Leftrightarrow w = s_1.s_2 \ldots s_n.\tau$ *for some* $\tau \in W_I$.

Proof. $\Rightarrow$: Since χ is dominant and $\tau \leq w_{0,I}$, for all $\tau \in W_I$, we have $\tau(\chi) \geq w_{0,I}(\chi)$; using the fact that $w_{0,I}(\alpha_i) = -\alpha_{n-i}$ for $i = 1, \cdots n-1$ and $w_{0,I}(\alpha_n) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ we have $w_{0,I}(\chi) = \sum_{i=1}^{n-1}(m_n - m_{n-i})\alpha_i + m_n.\alpha_n$. Therefore, $\tau(\chi) = \sum_{i=1}^{n-1} a_i\alpha_i + m_n\alpha_n$, $a_i > 0$. Now, let $w = \phi\tau$ with $\phi \in W^I$, $\tau \in W_I$. Therefore, $w(\chi) = \phi(\tau(\chi)) = \phi(\sum_{i=1}^{n-1} a_i\alpha_i + m_n\alpha_n)$. Thus $w(\chi) \leq 0$ implies that $\phi = s_1.s_2 \ldots s_n$.
$\Leftarrow$: Let $w = s_1.s_2 \ldots s_n.\tau$, $\tau \in W_I$. Now,

$$\begin{aligned} w(\chi) &= s_1.s_2 \ldots s_n\tau(\chi) \\ &= s_1.s_2 \ldots s_n(\textstyle\sum_{i=1}^{n-1} a_i\alpha_i + m_n\alpha_n) \\ &= -m_n\alpha_1 + \textstyle\sum_{i=2}^{n}(a_{i-1} - m_n)\alpha_i \end{aligned}$$

Since χ is a dominant weight we have $\chi - \tau(\chi) \geq 0$. Hence we have $a_i \leq m_i \leq m_n$. Thus $w(\chi) \leq 0$. This completes the proof. □

Consider $GL_n(k)$ as a subgroup of $GL_{n+1}(k)$ given by the inclusion $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let $B_n = B_{n+1} \bigcap GL_n(k)$ as a Borel subgroup with I as the simple roots.

Let χ be a regular dominant character as in Lemma (5.1).

Theorem 5.2. *We have an isomorphism*

$$\Psi : {}_{T}\backslash\backslash(GL_{n+1}(k)/B_{n+1}))^{ss}(L_\chi) \xrightarrow{\sim} GL_n(k)/B_n.$$

Proof. Proof uses cellular decomposition of both homogeneous spaces $GL_{n+1}(k)/B_{n+1}$ and $GL_n(k)/B_n$. First, we fix a total order on the set of positive roots of B_{n+1} such that $\sum_{i=1}^{n}\alpha_i > \sum_{i=1}^{n-1}\alpha_i > \cdots > \alpha_1 > \sum_{i=2}^{n}\alpha_i > \cdots > \alpha_2 > \sum_{i=3}^{n}\alpha_i > \cdots > \alpha_3 > \cdots > \alpha_{n-1}+\alpha_n > \alpha_n$. Now any GL_{n+1}/B_{n+1} (resp. GL_n/B_n) is the union of cells $U_w w B_{n+1}$ (resp. $U_\tau \tau B_n$) with $w \in W$ (resp. $\tau \in W_I$). Using the total order above we can write each element $x \in U_w$ as a product of u_α in the decreasing order from the left to the right. Let X_α (resp. Y_β) be the co-ordinate function on $U_w w B_{n+1}$ (resp. $U_\tau \tau B_n$) corresponding to the root α (resp. β).

With these notations we proceed the proof:

Let $\tau \in W_I$. Let $w := s_1 s_2 \cdots s_n \tau$. $V_\tau^0 : \{x = U_w w B_{n+1} : X_\alpha(x) \neq 0 \ \forall \ \alpha \geq \alpha_1\}$.

Set $V^0 := \bigcup_{\tau \in W_I} V_\tau^0$.

Step1: We prove that $(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \subset V^0$.

This can be seen as follows.

By Hilbert-Mumford criterion [see theorem 2.1 of [3]], a point $x \in GL_{n+1}/B_{n+1}$ is semi-stable $\Leftrightarrow \mu^L(x,\lambda) \geq 0$ for all 1- parameter subgroup λ of $T \Leftrightarrow \mu^L(\sigma x, \lambda) \geq 0$ for all one parameter subgroups $\lambda \in \overline{C(B)}$ and for all $\sigma \in W$. By the lemma 2.1 of [6], this statement is equivalent to $\langle -w_\sigma \chi, \lambda \rangle \geq 0$ for all $\lambda \in \overline{C(B)}$ where $\sigma x \in U_{w_\sigma} w_\sigma B$. But this is equivalent to $w_\sigma(\chi) \leq 0$. And this is equivalent to w_σ is of the form $(s_1 \ldots s_n).\tau_1$ for some $\tau_1 \in W_I$. Now let $x \in U_w w B_{n+1}$ with $w = (s_1 \ldots s_n)\tau$, $\tau \in W_I$.

Now, let $X_\alpha(x) = 0$ for some $\alpha \geq \alpha_1$. Let $\alpha = \sum_{j=1,\cdots i}\alpha_j$. Then, we have $s_1 s_2 \cdots s_i x = u'\phi B_{n+1}$ with $\phi \neq s_1 \cdots s_n \tau$ for any $\tau \in W_I$. Hence, by the above discussion, x is not semi-stable.

Step 2: $(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) = V^0$. This can be seen by the above discussion and from the following *claim*.

claim: V is W-stable.

Proof of *claim*: Let $\tau \in W_I$. Let $x \in U_{s_1 s_2 \cdots s_n \tau} s_1 s_2 \cdots s_n \tau B_{n+1}$, with $X_\alpha(x) \neq 0$ for all $\alpha \geq \alpha_1$. Then, we have $s_1 x \in U_{s_1 s_2 \cdots s_n \tau} B_{n+1}$ with $X_\alpha(s_1 x) = -\frac{X_\alpha(x)}{X_{\alpha_1}(x)}$ for $\alpha > \alpha_1$, and $X_{\alpha_1}(s_1 x) = \frac{1}{X_{\alpha_1}(x)}$. Hence, $s_1 x \in V^0$.

Now, let $i \neq 1$. If $X_{\alpha_i}(u) = 0$, then, $s_i x = u' s_1 s_2 \cdots s_n s_{i-1} \tau B_{n+1}$ with $X_\alpha(s_1 x) = X_{s_{i(\alpha)}}(x)$. Hence, $s_i(x) \in V^0$. Otherwise, we must have $s_i x \in U_{s_1 s_2 \cdots s_n \tau} B_{n+1}$ with $X_\alpha(s_i x) = X_\alpha(x)$ for all such that $s_i(\alpha) = \alpha$, $X_\alpha(s_i x) = \frac{X_\alpha(x)}{X_{\alpha_i}(x)}$ for all α of the form $\alpha = \sum_{j=k}^{i}\alpha_j$ such that $k < i$, $X_{\alpha_i}(s_i x) = \frac{1}{X_{\alpha_i}(x)}$, and $X_\alpha(s_i x) = \frac{-X_\alpha(x)}{X_{\alpha_i}(x)}$ for all α of the form $\alpha = \sum_{j=i}^{k}\alpha_j$ such that $k > i$.

Hence $s_i V^0 \subset V^0$ for all $i = 1, \cdots n$. Thus, the *claim* follows from the fact that W is

generated by s_i's.

Step 3: Now, for each $\tau \in W_I$, we exhibit an isomorphism

$$\Psi_\tau :_{T\backslash\backslash} V^0_\tau \stackrel{\sim}{\longrightarrow} U_\tau \tau B_n/B_n.$$

Let $\tau \in W_I$, consider the map $\pi_\tau : V^0_\tau \longrightarrow (U_\tau \tau B_n)/B_n$ defined by $\phi_\tau(x) = y$ with for each $\beta \ngeq \alpha_1$ $Y_{s_n \ldots s_1(\beta)}(y) = (\frac{-X_\beta(x)X_{\beta'}(x)}{X_{\beta+\beta'}(x)})$ where for each $\beta \in R^+(w^{-1})$ with $\beta \ngeq \alpha_1$, β' is the unique element of R^+ with $\beta' \geq \alpha_1$ such that $\beta + \beta' \in R^+$. Clearly this map is T-invariant. Thus the morphism π_τ give rise to a morphism

$$\Psi_\tau :_{T\backslash\backslash} V^0_\tau \longrightarrow U_\tau \tau B_n/B_n.$$

Clearly $\Psi_\tau$is surjective. We now prove that Ψ_τ is injective:

π_w is injective for each $w \in W$ of the form $w = s_1.s_2 \ldots s_n\tau$, for some $\tau \in W_I$. Let x_1 and x_2 be two points of V^0_τ such that

$\pi_\tau(x_1) = \pi_\tau(x_2)$. Hence, $\frac{X_\beta(x_1)X_{\beta'}(x_1)}{X_{\beta+\beta'}(x_1)} = \frac{X_\beta(x_2)X_{\beta'}(x_2)}{X_{\beta+\beta'}(x_2)}$. Let $t \in T$ be such that $(\alpha_1 + \cdots + \alpha_i)(t) = \frac{X_{\alpha_1+\cdots+\alpha_i}(x_2)}{X_{\alpha_1+\cdots+\alpha_i}(x_1)}$ for all i, $1 \leq i \leq n$. Then, it is easy to check that $t \cdot x = y$.
Thus Ψ_τ is bijective for each $\tau \in W_I$.

Step 4: Ψ_τ puts together to give an isomorphism

$$\Psi :_{T\backslash\backslash} V^0 \stackrel{\sim}{\longrightarrow} GL_n(k)/B_n.$$

Since the W- translates of $V^0_{w_{0,I}}$ is the whole of V^0, and W_I- translates of $U_{w_{0,I}}w_{0,I}B_n$ is the whole of GL_n/B_n, and there is an isomorphism from $W_{S\setminus\{\alpha_1\}}$ to W_I taking s_i to s_{i-1} for each $i = 2, \cdots n$, to prove the Theorem, it is sufficient to prove that the T - invariant morphisms $\pi_\tau : V^0_\tau \longrightarrow U_\tau\tau B_n$, and $\pi_{s_{i-1}\tilde{}\tau}U_{s_{i-1}\tilde{}\tau} \longrightarrow U_{s_{i-1}\tilde{}\tau}$ satisfy the following:

$Y_\alpha(\pi_\tau(x)) = Y_\alpha(s_{i-1}(\pi_{s_{i-1}\tilde{}\tau}(s_i x)))$ for each $\alpha \in R^+(\tau^{-1})$. (Here, the notation $s_{i-1}\tilde{}\tau = \tau$ if $s_{i-1}\tau < \tau$, and $s_{i-1}\tilde{}\tau = s_{i-1}\tau$ otherwise.)

We make use of the following observations using commutator relations:

$$X_\alpha(s_i; x) = \begin{cases} \frac{-X_\alpha(x)}{X_{\alpha_i}(x)} & \text{if } \alpha = \alpha_i + \cdots \alpha_k,\ i < k \ \text{ and } \ w^{-1}(\alpha_{i+1} + \cdots \alpha_k) > 0, \\ \frac{1}{X_{\alpha_i}(x)} & \text{if } \alpha = \alpha_i, \\ X_{s_i(\alpha)}(x) & otherwise \end{cases}.$$

Let $\alpha \in R^+(\tau^{-1})$

Case 1: $\alpha = \alpha_{k-1} + \cdots \alpha_{i-1}$, $k < i$, $w^{-1}(\alpha_k + \cdots \alpha_i) = \tau^{-1}(\alpha_{k-1} + \cdots \alpha_{i-1}) > 0$ and $s_{i-1}\tilde{\tau} = \tau$.

In this case, $Y_\alpha(s_{i-1}(\pi_\tau(x))) = \frac{X_{\alpha_1+\cdots\alpha_{k-1}}(x)X_{\alpha_k+\cdots+\alpha_i}(x)}{X_{\alpha_1+\cdots\alpha_{i-1}}(x)} = Y_\alpha(\pi_\tau(s_i x))$.

Case 2: $\alpha = \alpha_{i-1}+\cdots\alpha_{k-1}$, $i<k$ and $w^{-1}(\alpha_i+\cdots\alpha_k) = \tau^{-1}(\alpha_{i-1}+\cdots\alpha_{k-1}) > 0$ and $s_{i-1}\tilde{\tau} = \tau$.

In this case, $Y_\alpha(s_{i-1}(\pi_\tau(x))) = -\frac{X_{\alpha_1+\cdots\alpha_i}(x)X_{\alpha_i+\cdots+\alpha_k}(x)}{X_{\alpha_i}(x)X_{\alpha_1+\cdots\alpha_k}(x)} = Y_\alpha(\pi_\tau(s_i x))$.

Case 3: $\alpha = \alpha_{i-1}$. $Y_\alpha(s_{i-1}(\pi_\tau(x))) = \frac{X_{\alpha_1+\cdots\alpha_i}(x)}{X_{\alpha_1+\cdots\alpha_{i-1}}(x)X_{\alpha_i}(x)} = Y_\alpha(\pi_\tau(s_{i-1}(x)))$.

In all other cases, we have: $Y_\alpha(s_{i-1}(\pi_{s_{i-1}\tilde{\tau}}(s_i x))) = \frac{X_{s_i s_1\cdots s_n(\alpha)}(x)X_{s_i(\beta')}(x)}{X_{s_i(s_1\cdots s_n(\alpha)+\beta')}(x)} = Y_\alpha(\pi_\tau(x))$, where β' is the unique root such that $\beta' \geq \alpha_1$ and $s_1\cdots s_n(\alpha)+\beta'$ is a root.

This completes the proof. $\square$

With Y_α's as in the proof of theorem 5.2, we have

Corollary 5.3.

$$s_1(Y_\alpha) = \begin{cases} -(1+Y_\alpha) & \textit{if } \alpha \geq \alpha_1. \\ Y_\alpha & \textit{otherwise} \end{cases} .$$

Proof. Proof follows from the fact that

$$X_\alpha(s_1 x) = \begin{cases} X_{\alpha_1}X_\alpha(x) + X_{\alpha_1+\alpha}(x) & \text{if } \alpha = \alpha_2+\cdots\alpha_i,\ 2\leq i, \\ \frac{-X_\alpha(x)}{X_{\alpha_1}(x)} & \text{if } \alpha = \alpha_1+\cdots\alpha_i,\ i\geq 2, \\ X_\alpha(x) & \text{if } \alpha = \alpha_3+\cdots\alpha_i,\ i\geq 3 \end{cases} .$$

$\square$

Corollary 5.4. *Let $\mathfrak{h}_n$ be a Cartan subalgebra of $sl_{n+1}(k)$. Let χ be a regular dominant character as in Theorem 5.2. Then, the action of W on the GIT quotient*

$$_T\backslash\backslash(GL_{n+1}(k)/B_{n+1}))^{ss}(L_\chi) \simeq GL_n(k)/B_n$$

is given by the n- dimensional representation $\mathfrak{h}_n$ of W.

Proof. Proof follows from theorem 5.2 and corollary 5.3. $\square$

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